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801 Homework 3

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Problem 1:

Let $y_{11}, y_{12}, ..., y_{1r}$ be $N(\mu_1, \sigma^2)$ and $y_{21}, y_{22}, ..., y_{2s}$ be $N(\mu_2, \sigma^2)$ with all y_{ij}'s independent. Write this as a linear model. Find estimates of $\mu_1, \mu_2, \mu_1 - \mu_2$, and σ^2. Using Appendix E, and Exercise 2.1, form an $\alpha = .01$ test for $H_0\colon \mu_1 = \mu_2$. Similarly, form 95% confidence intervals for $\mu_1 - \mu_2$ and μ_1. What is the test for $H_0\colon \mu_1 = \mu_2 + \Delta$, where Δ is some known fixed quantity? How do these results compare with the usual analysis for two independent samples?

Solution: Writing this as linear model, we have

$$Y = X\beta + \epsilon,$$

where

$$Y = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{bmatrix}_{(r+s)\times 2}, \quad X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}_{(r+s)\times 2}, \quad \beta = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{r+s} \end{bmatrix},$$

and $\epsilon \sim N_{r+s}(0, \sigma^2 I)$. We now find estimates of μ_1, μ_2, and $\mu_1 - \mu_2$. First, note that the o.p.m M onto $C(X)$ is

$$M = X(X'X)^- X' = \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{r}}_{r\times r} & \mathbf{0}_{r\times s} \\ \mathbf{0}_{s\times r} & \frac{\mathbf{1}}{\mathbf{s}}_{s\times s} \end{bmatrix},$$

where $\frac{\mathbf{1}}{\mathbf{r}}_{r\times r}$ represents an $r \times r$ matrix of all entries $\frac{1}{r}$ and similar idea for $\frac{\mathbf{1}}{\mathbf{s}}_{s\times s}$. Also notice that if ρ' is the matrix

$$\rho' = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{bmatrix}_{3\times(r+s)},$$

then

$$\rho' X\beta = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_1 - \mu_2 \end{bmatrix}.$$

By Corollary 2.2.3, the LSE of $\rho' X\beta$ is

$$\rho' MY = \begin{bmatrix} \frac{1}{r} & \cdots & \frac{1}{r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{s} & \cdots & \frac{1}{s} \\ \frac{1}{r} & \cdots & \frac{1}{r} & -\frac{1}{s} & \cdots & -\frac{1}{s} \end{bmatrix} Y = \begin{bmatrix} \frac{1}{r}\sum_{i=1}^{r} y_{1i} \\ \frac{1}{s}\sum_{i=1}^{s} y_{2i} \\ \frac{1}{r}\sum_{i=1}^{r} y_{1i} - \frac{1}{s}\sum_{i=1}^{s} y_{2i} \end{bmatrix} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_1 - \overline{y}_2 \end{bmatrix}.$$

Therefore, an estimate of μ_1 is $\widehat{\mu}_1 = \overline{y}_1$, an estimate of μ_2 is $\widehat{\mu}_2 = \overline{y}_2$, and an estimate of $\mu_1 - \mu_2$ is $\overline{y}_1 - \overline{y}_2$. Now we find an estimate of σ^2. Since $r(X) = 2$ and $\mathrm{Cov}(\epsilon) = \sigma^2 I$, then by theorem 2.2.6,

$$MSE = \frac{Y'(I-M)Y}{(r+s)-2} = \frac{\sum_{i=1}^{r}(y_{1i}-\overline{y}_1)^2 + \sum_{i=1}^{s}(y_{2i}-\overline{y}_2)^2}{(r+s)-2}$$

is an estimate of σ^2. Now we form an $\alpha = .01$ test for $H_0\colon \mu_1 = \mu_2$. First, rewrite the null hypothesis as $H_0\colon \mu_1 - \mu_2 = 0$ and note by exercise 2.1 and Appendix E

$$\frac{\lambda'\widehat{\beta} - \lambda'\beta}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \sim t\,(1-\alpha/2, dfE)\,.$$

Now if $\lambda' = \begin{bmatrix} 1 & -1 \end{bmatrix}$, then under the null hypothesis,

$$T = \frac{\overline{y}_1 - \overline{y}_2 - 0}{\sqrt{MSE\left(\frac{1}{r} + \frac{1}{s}\right)}}$$

should be an observation from $t(1-\alpha/2, r+s-2)$. Therefore, reject H_0 if $|T| \geq t(.995, (r+s)-2)$. Also by Appendix E, a 95% confidence interval for $\mu_1 - \mu_2$ is

$$\left[\overline{y}_1 - \overline{y}_2 \pm t(.975, (r+s)-2)\sqrt{MSE\left(\frac{1}{r} + \frac{1}{s}\right)}\right].$$

Also a 95% confidence interval for μ_1 is

$$\left[\overline{y}_1 \pm t(.975, (r+s)-2)\sqrt{MSE \cdot \frac{1}{r}}\right].$$

Lastly, we develop the test for $H_0\colon \mu_1 = \mu_2 + \Delta$. Similarly, we construct the statistic

$$T = \frac{\overline{y}_1 - \overline{y}_2 - \Delta}{\sqrt{MSE\left(\frac{1}{r} + \frac{1}{s}\right)}}$$

where

$$MSE = \frac{\sum_{i=1}^{r}(y_{1i}-\overline{y}_1)^2 + \sum_{i=1}^{s}(y_{2i}-\overline{y}_2)^2}{r+s-2} = \frac{(r-1)s_1^2 + (s-1)s_2^2}{r+s-2}.$$

Reject H_0 if $|T| \geq t(1-\alpha/2, (r+s)-2)$. Therefore, these results are the same as the usual analysis for two independent samples.

Problem 2:

Let $y_1, y_2, ..., y_n$ be independent $N(\mu, \sigma^2)$. Write a linear model for these data. For the rest of the problem, use the results of Chapter 2, Appendix E, and Exercise 2.1. Form an $\alpha = 0.01$ test for $H_0\colon \mu = \mu_0$, where μ_0 is some known fixed number and form a 95% confidence interval for μ. How do these results compare with the usual analysis for one sample?

Solution: Writing this as a linear model, we have

$$Y = X\beta + \epsilon,$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}, \quad \beta = \begin{bmatrix} \mu \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

and $\epsilon \sim N_n(0, \sigma^2 I)$. We showed in problem 1 that an estimate of μ is $\widehat{\mu} = \overline{y}$. Note that the projection matrix is

$$M = X(X'X)^{-}X' = \left[\tfrac{\mathbf{1}}{\mathbf{n}}\right]_{n\times n},$$

i.e. M is an $n \times n$ matrix of all entries $\frac{1}{n}$, and so the MSE becomes

$$MSE = \frac{Y'(I-M)Y}{n-1} = \frac{\sum_{i=1}^{n}(y_i - \overline{y})^2}{n-1} = s^2.$$

Then, taking $\lambda' = 1$, we know that under the null hypothesis

$$T = \frac{\overline{y} - \mu_0}{\sqrt{MSE(\frac{1}{n})}} = \frac{\overline{y} - \mu_0}{\sqrt{s^2/n}}$$

should be an observation from $t(.995, n-1)$. Therefore, reject H_0 if $|T| \geq t(.995, n-1)$. A 95% confidence interval for μ is

$$\left[\overline{y} \pm t(.975, n-1)\sqrt{MSE \cdot \frac{1}{n}}\right].$$

These results coincide with the usual analysis for one sample.

Problem 3:

(a) Show that $AVA' = AV = VA'$.

(b) Show that $A'V^{-1}A = A'V^{-1} = V^{-1}A$.

(c) Show that A is the same for any choice of $(X'V^{-1}X)^{-}$.

Solution: For this problem, $A = X(X'V^{-1}X)^{-}X'V^{-1}$, where V is a covariance matrix and hence is symmetric.

(a) It is clear that $AV = VA'$ because

$$AV = X(X'V^{-1}X)^{-}X'$$

and

$$VA' = V(V^{-1}X(X'V^{-1}X)^{-}X') = X(X'V^{-1}X)^{-}X' = AV.$$

Now,

$$\begin{aligned} AVA' &= \Big(X(X'V^{-1}X)^{-}X'V^{-1}\Big)V\Big(X(X'V^{-1}X)^{-}X'V^{-1}\Big)' \\ &= X(X'V^{-1}X)^{-}(X'V^{-1}X)(X'V^{-1}X)^{-}X' \\ &= X(X'V^{-1}X)^{-}X' \\ &= AV = VA'. \end{aligned}$$

Therefore, $AVA' = AV = VA'$.

(b) It is clear that $A'V^{-1} = V^{-1}A$ because

$$A'V^{-1} = V^{-1}X(X'V^{-1}X)^{-}X'V^{-1}$$

and

$$V^{-1}A = V^{-1}X(X'V^{-1}X)^{-}X'V^{-1} = A'V^{-1}.$$

Now,

$$\begin{aligned} A'V^{-1}A &= \Big(X(X'V^{-1}X)^{-}X'V^{-1}\Big)'V^{-1}\Big(X(X'V^{-1}X)^{-}X'V^{-1}\Big) \\ &= V^{-1}X(X'V^{-1}X)^{-}(X'V^{-1}X)(X'V^{-1}X)^{-}X'V^{-1} \\ &= V^{-1}X(X'V^{-1}X)^{-}X'V^{-1} \\ &= A'V^{-1} = V^{-1}A. \end{aligned}$$

Therefore, $A'V^{-1}A = A'V^{-1} = V^{-1}A$.

(c) Note that since V is positive definite, we can write $V^{-1} = Q'Q$ for some Q. Then,

$$X'V^{-1}X = X'Q'QX = (QX)'(QX) := (X^\star)'X^\star.$$

Now assume that G and H are generalized inverses of $(X'V^{-1}X)$, i.e. of $(X^\star)'X^\star$. Then, by Lemma B.43,

$$\begin{aligned}
X^\star G(X^\star)' &= X^\star H(X^\star)' \\
QXGX'Q' &= QXHX'Q' \\
QXGX'Q'Q &= QXHX'Q'Q \\
QXGX'V^{-1} &= QXHX'V^{-1} \\
Q'QXGX'V^{-1} &= Q'QXHX'V^{-1} \\
V^{-1}XGX'V^{-1} &= V^{-1}XHX'V^{-1} \\
XGX'V^{-1} &= XHX'V^{-1}.
\end{aligned}$$

Now, recall that $A = X(X'V^{-1}X)^{-}X'V^{-1}$. Then, we have $A = XGX'V^{-1} = XHX'V^{-1}$. Therefore, A is the same for any choice of $(X'V^{-1}X)^{-}$.

Problem 4:

Consider the model

$$Y = X\beta + b + e, \qquad E(e) = 0, \qquad \mathrm{Cov}(e) = \sigma^2 I,$$

where b is a known vector. Show that Proposition 2.1.9: A linear estimate $a_0 + a'Y$ is unbiased for $\lambda'\beta$ if and only if $a_0 = 0$ and $a'X = \lambda'$. is not valid for this model by producing a linear unbiased estimate of $\rho'X\beta$, say $a_0 + a'Y$, for which $a_0 \neq 0$. Hint: Modify $\rho'MY$.

Solution: We know that

$$E(a_0 + a'Y) = a_0 + a'E(Y) = a_0 + a'(X\beta + b).$$

Then, take $a_0 = -a'b$ and $a' = \rho'$. This gives that

$$E(a_0 + a'Y) = a_0 + a'X\beta + a'b = a'X\beta = \rho'X\beta,$$

that is to say that $a_0 + a'Y$ is unbiased for $\rho'X\beta$, but $a_0 \neq 0$. Therefore, proposition 2.1.9 is not valid.

Problem 5:

Consider the model $y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + e_i$, where $e_i \overset{iid}{\sim} N(0, \sigma^2)$. Use the data given below to answer (a) through (d).

(a) Estimate β_1, β_2, and σ^2.

(b) Give 95% confidence intervals for β_1 and $\beta_1 + \beta_2$.

(c) Perform an $\alpha = .01$ test for $H_0\colon \beta_2 = 3$.

(d) Find an appropriate P value for the test of $H_0\colon \beta_1 - \beta_2 = 0$.

Solution: First, note that by the data given, we have

$$Y = \begin{bmatrix} 82 \\ 79 \\ 74 \\ 83 \\ 80 \\ 81 \\ 84 \\ 81 \end{bmatrix}, \quad X = \begin{bmatrix} 10 & 15 \\ 9 & 14 \\ 9 & 13 \\ 11 & 15 \\ 11 & 14 \\ 10 & 14 \\ 10 & 16 \\ 12 & 13 \end{bmatrix}.$$

(a) Let $\beta = (\beta_1, \beta_2)'$. Then, we know the LSE of β is

$$\widehat{\beta} = (X'X)^{-}X'Y = \begin{bmatrix} 2.65 \\ 3.74 \end{bmatrix}.$$

Also, we know the LSE of σ^2 is

$$MSE = \frac{Y'(I-M)Y}{n-r},$$

where $M = X(X'X)^{-}X'$ and $r = r(X)$. Therefore,

$$\widehat{\sigma}^2 = MSE = \frac{Y'(I-M)Y}{8-2} = 4.70.$$

Therefore, our estimates are $\widehat{\beta_1} = 2.65$, $\widehat{\beta_2} = 3.74$, and $\widehat{\sigma}^2 = 4.70$.

(b) Let $\lambda_1 = (1, 0)$ and $\lambda_2 = (1, 1)'$. Note that

$$(X'X)^{-} = \frac{1}{19712}\begin{bmatrix} 1632 & -1168 \\ -1168 & 848 \end{bmatrix}.$$

Then, a 95% confidence interval for β_1 is

$$\left[\widehat{\beta_1} \pm t(.975, 8-2)\sqrt{MSE \cdot \lambda_1'(X'X)^{-}\lambda_1}\right] = [1.124, 4.176].$$

Also, a 95% confidence interval for $\beta_1 + \beta_2$ is

$$\left[\widehat{\beta_1} + \widehat{\beta_2} \pm t(.975, 8-2)\sqrt{MSE \cdot \lambda_2'(X'X)^{-}\lambda_2}\right] = [5.937, 6.843].$$

(c) Take $\lambda = (0,1)'$. Then, we calculate the test statistic

$$T = \frac{\widehat{\beta}_2 - 3}{\sqrt{MSE \cdot \lambda'(X'X)^{-}\lambda}} = \frac{3.74 - 3}{\sqrt{4.70 \cdot \frac{848}{19712}}} = 1.646.$$

Since $|T| \ngeq t(.995, 6) = 3.707$, we fail to reject H_0.

(d) Take $\lambda = (1, -1)$. Then, we calculate the test statistic

$$T = \frac{\widehat{\beta}_1 - \widehat{\beta}_2 - 0}{\sqrt{MSE \cdot \lambda'(X'X)^{-}\lambda}} = \frac{-1.09}{\sqrt{4.70 \cdot \frac{4816}{19712}}} = -1.017.$$

Then, the P value is $P = P(|T| \geq 1.017) = 2P(T < -1.017) = 0.348$.

Problem 6:

Consider the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{11} x_{i1}^2 + \beta_{22} x_{i2}^2 + \beta_{12} x_{i1} x_{i2} + e_i,$$

where the predictor variables take on the values given. Show that $\beta_0, \beta_1, \beta_2, \beta_{11} + \beta_{22}$, and β_{12} are estimable and find (nonmatrix) algebraic forms for the estimates of these parameters. Find the MSE and the standard errors of the estimates.

Solution: For this model, we have

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{bmatrix}.$$

It will be useful to identify a basis for $C(X')$. Notice that the row reduced form of X is

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, a basis for $C(X')$ are the first five rows of R. Denote these rows as r_1, r_2, r_3, r_4, and r_5, respectively.
Take $\Lambda_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$. Then,

$$\Lambda_0' \beta = \beta_0$$

and $\Lambda_0 = r_1 - r_2 - r_3 - r_4$. Therefore, $\Lambda_0 \in C(X')$ and so β_0 is estimable.
Take $\Lambda_1 = \begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}'$. Then,

$$\Lambda_1'\beta = \beta_1$$

and $\Lambda_1 = r_2 - r_5$. Therefore, $\Lambda_1 \in C(X')$ and so β_1 is estimable.
Take $\Lambda_2 = \begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}'$. Then,

$$\Lambda_2'\beta = \beta_2$$

and $\Lambda_2 = r_3$. Therefore, $\Lambda_2 \in C(X')$ and so β_2 is estimable.
Take $\Lambda_3 = \begin{bmatrix}0 & 0 & 0 & 1 & 1 & 0\end{bmatrix}'$. Then,

$$\Lambda_3'\beta = \beta_{11} + \beta_{22}$$

and $\Lambda_3 = r_4$. Therefore, $\Lambda_3 \in C(X')$ and so $\beta_{11} + \beta_{22}$ is estimable.
Lastly, take $\Lambda_4 = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}'$. Then,

$$\Lambda_4'\beta = \beta_{12}$$

and $\Lambda_4 = r_5$. Therefore, $\Lambda_4 \in C(X')$ and so β_{12} is estimable. To calcualte the estimates, we first calculate a generalized inverse to be

$$(X'X)^- = \frac{1}{48}\begin{bmatrix}16 & 0 & 0 & -8 & -8 & 0\\ 0 & 12 & 0 & 0 & 0 & 0\\ 0 & 0 & 12 & 0 & 0 & 0\\ -8 & 0 & 0 & 7 & 7 & 0\\ -8 & 0 & 0 & 7 & 7 & 0\\ 0 & 0 & 0 & 0 & 0 & 12\end{bmatrix}.$$

Then, the estimates can be found by using the normal equations, which gives

$$\widehat{\beta} = (X'X)^- X'Y = \begin{bmatrix}\frac{1}{3}(y_5 + y_6 + y_7)\\ \frac{1}{4}(y_1 + y_2 - y_3 - y_4)\\ \frac{1}{4}(y_1 - y_2 + y_3 - y_4)\\ \frac{1}{8}(y_1 + y_2 + y_3 + y_4) - \frac{1}{6}(y_5 + y_6 + y_7)\\ \frac{1}{8}(y_1 + y_2 + y_3 + y_4) - \frac{1}{6}(y_5 + y_6 + y_7)\\ \frac{1}{4}(y_1 - y_2 - y_3 + y_4)\end{bmatrix} = \begin{bmatrix}\widehat{\beta}_0\\ \widehat{\beta}_1\\ \widehat{\beta}_2\\ \widehat{\beta}_{11}\\ \widehat{\beta}_{22}\\ \widehat{\beta}_{12}\end{bmatrix}.$$

Then, we have the estimate of $\widehat{\beta}_{11} + \widehat{\beta}_{22} = \frac{1}{4}(y_1 + y_2 + y_3 + y_4) - \frac{1}{3}(y_5 + y_6 + y_7)$.
To find the MSE, we must first find $(I - M)$. We calculate M to be

$$M = X(X'X)^- X' = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}.$$

Then, we get $(I - M)$ to be

$$I - M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Therefore, we have that MSE is

$$MSE = \frac{Y'(I - M)Y}{7 - 5} = \frac{1}{3}(y_5^2 - y_5y_6 - y_5y_7 + y_6^2 - y_6y_7 + y_7^2).$$

Lastly, we can calculate the SE of these estimates by

$$SE = \sqrt{MSE\lambda'(X'X)^{-}\lambda}.$$

Using the lambdas when showing estimability, we have

$$\begin{aligned} SE\left(\widehat{\beta}_0\right) &= \sqrt{MSE \cdot \Lambda_0'(X'X)^{-}\Lambda_0} = \sqrt{MSE \cdot \frac{1}{3}} \\ SE\left(\widehat{\beta}_1\right) &= \sqrt{MSE \cdot \Lambda_1'(X'X)^{-}\Lambda_1} = \sqrt{MSE \cdot \frac{1}{4}} \\ SE\left(\widehat{\beta}_2\right) &= \sqrt{MSE \cdot \Lambda_2'(X'X)^{-}\Lambda_2} = \sqrt{MSE \cdot \frac{1}{4}} \\ SE\left(\widehat{\beta}_{11} + \widehat{\beta}_{22}\right) &= \sqrt{MSE \cdot \Lambda_3'(X'X)^{-}\Lambda_3} = \sqrt{MSE \cdot \frac{7}{12}} \\ SE\left(\widehat{\beta}_{12}\right) &= \sqrt{MSE \cdot \Lambda_4'(X'X)^{-}\Lambda_4} = \sqrt{MSE \cdot \frac{1}{4}}. \end{aligned}$$